Math 53, Discussions 116 and 118 **2022-8-26**

Parametric curves and their tangents

Answers included

Conceptual questions

Question 1. In lecture, you saw that $x = \cos t, y = \sin t$ parametrizes a circle. Where is the center of the circle, and what is the circle's radius? What point on the circle corresponds to $t = 0$? As t increases, is the circle traced clockwise or counterclockwise? How long does it take to trace out the circle exactly once?

Then answer the same questions for these other circles:

- $x = 3\sin t, y = 3\cos t$
- $x = \cos(2t), y = \sin(2t)$
- $x = 4 + \cos t, y = -3 + \sin t$

Question 2. Beware: the following parametric curves are *not* circles. (What are they?)

- $x = 2\cos t, y = 5\sin t$
- $x = \sin t, y = \cos(2t)$. (Hint: use the double angle formula to find a Cartesian equation.)

Question 3. Suppose that a parametric curve $x = f(t), y = g(t)$ satisfies $g'(3) = 0$. What can you conclude (if anything) about the tangent to the curve at $t = 3$?

Computations

Problem 1. This is a further exploration of some ideas from Question 1, so that you can better understand how to transform shapes algebraically. Let C be the curve described by the equation $x^2 + y^2 = 5$. If we take C and

(1) shift it by 2 units in the positive y direction (i.e. upwards),
(2) and then stretch it by a factor of 3 in the x direction (i.e. horizontally),

what Cartesian equation describes the resulting shape?

Next, come up with a parametrization $x = f(t), y = g(t)$ for the starting shape C, and then a parametrization for the shape obtained after applying the transformations.

Problem 2. Find a Cartesian equation for the parametric curve $x = t^3 + t, y = t^2 + 2$. Hint: compute x^2.

Problem 3. There are two points on the curve

$$x = 2t^2, y = t - t^2, -\infty < t < \infty$$

where the tangent line passes through the point $(10, -2)$. Find these two points.

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1.

- Center $(0,0)$, radius 1, start at $(1,0)$, complete one CCW revolution in 2π time.
- Center $(0,0)$, radius 3, start at $(0,3)$, complete one CW revolution in 2π time.
- Center $(0,0)$, radius 1, start at $(1,0)$, complete one CCW revolution in π time.
- Center $(4,-3)$, radius 1, start at $(5,-3)$, complete one CCW revolution in 2π time.

Question 2.

- This is an ellipse. I drew a picture of it in discussion.
- By eliminating the parameter t using the double angle formula, one finds the Cartesian equation $y = 1 - 2x^2$. But beware: this parametric curve is not the entire parabola described by the Cartesian equation: only the portion between the points $(-1,-1)$ and $(1,-1)$.

Question 3. From the information given, you cannot conclude anything about the tangent line in question. If you moreover knew that $f'(3) \neq 0$, then it would follow that the tangent line is horizontal. But without that knowledge, the slope of the tangent line could be anything. (In the event that both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are zero, you need to compute a limit to find the slope of the tangent line.)

Answers to computations

Problem 1. We did this one together in discussion. The summary is that, to shift a shape by 2 units in the positive y direction, replace y by $y-2$. Stretching by a factor of 3 in the x direction is achieved by replacing x by $x/3$.

Problem 2. Following the hint: we get $x^2 = t^6 + 2t^4 + t^2$. Since $t^2 = y - 2$, we can eliminate t and obtain the equation

$$x^2 = (y-2)^3 + 2(y-2)^2 + (y-2).$$

Problem 3. The tangent line at the time t has slope given by

$$\frac{dy}{dx} = \frac{1-2t}{4t}$$

The point corresponding to the time t on the curve is none other than $(2t^2, t - t^2)$, so by the point-slope formula we have

$$y - (t - t^2) = \frac{1-2t}{4t}(x - 2t^2)$$

as the equation of the tangent line at time t for all times $t \neq 0$. When $t = 0$ we have a vertical tangent and I leave it to you to check that it does not pass through the point $(10,-2)$.

The equation we need to solve is

$$-2 - (t - t^2) = \frac{1-2t}{4t}(10 - 2t^2)$$

because we want to find the times t for which the tangent line passes through the point $(10,-2)$. This simplifies into a quadratic equation in t with solutions $t = 1, 5$, corresponding to the points $(2,0)$ and $(50,-20)$ respectively.